# Elements of Partial Differential Equations

## Preliminary facts

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# Contents

# Chapter 1

# Preliminary chapter.

## 1.1 Introduction

In this Chapter we shall introduce two basic tools for the study of partial differential equations (PDE). Namely, we start with the Fourier transform. We shall avoid a complete and detailed representation of the theory of Fourier transform and Sobolev spaces. Nevertheless, we shall underline only the points, which are important for a further study of PDE.

The reader can use [45], [22] , [44] [6] for more detailed information about the space of distributions, Fourier transform and the convolution.

## 1.2 Gauss - Green formula for domains with smooth boundaries in $\mathbb{R}^n$.

Our goal is to recall the classical Gauss Green formula valid for any open domain $\Omega$ in $\mathbb{R}^n$ with smooth (or $C^2$) boundary $\partial\Omega$. Given any $n$ continuous function

$$F(x) = (f_1(x), \cdots, f_n(x))$$

in the closure of $\Omega$ such that all first derivatives of $f_k(x)$ are well defined and continuos one has the formula

$$\int_\Omega \nabla.F(x)dx = \int_{\partial\Omega} N.F(x)dS_x, \tag{1.2.1}$$

where $F(x) = (f_1(x), \cdots, f_n(x))$,

$$\nabla.F = \partial_1 f_1(x) + \partial_2 f_2(x) + \cdots + \partial_n f_n(x).$$

Recall also that $N(x)$ is the unit outern normal at $x \in \partial\mathbf{O}$, while $dS_x$ is the surface element on $\partial\mathbf{O}$.

Recall that for any surface $\partial\mathbf{O} \subset \mathbf{R}^n$ defined (even locally) by

$$\partial\Omega: \; x_n = \psi(x'), x' = (x_1, ..., x_{n-1}) \in \mathbf{R}^{n-1} \tag{1.2.2}$$

one can define the corresponding surface element

$$dS_x = \sqrt{1 + |\nabla\psi(x')|^2}\, dx'.$$

**Remarks on Riemannian metric associated with the surface**

The Riemannian metric

$$dl^2 = (dx')^2 + dx_n^2 \tag{1.2.3}$$

induces on $\partial\Omega$ a Riemannian metric. More precisely, the metric induced by the embedding $\partial\Omega \subset \mathbf{R}^n$ is

$$ds^2 = (dl_{|\partial\Omega})^2 = \sum_{i,j=1}^{n-1} g_{ij}\, dx^i dx^j, \tag{1.2.4}$$

where

$$g_{ij} = \delta_{ij} + \frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}. \tag{1.2.5}$$

**Problem 1.2.1.** *Prove that*

$$g = \det(g_{ij}) = 1 + |\nabla\psi|^2.$$

The unit vector normal $N(x)$ to $\partial\mathbf{O}$ at the point $(x, \psi(x)) \in S$ is

$$N(x) = \frac{(1, -\nabla\psi(x))}{\sqrt{1 + |\nabla\psi|^2}}.$$

The second quadratic form on $\partial\mathbf{O}$ is

$$\sum_{i,j=1}^{n} b_{ij}\, dx^i dx^j,$$

where

$$b_{ij}(x) = \frac{\partial_{x_i}\partial_{x_j}\psi(x)}{\sqrt{1 + |\nabla\psi|^2}}.$$

Now the Gauss curvature $K(x)$ is

$$K(x) = \frac{\det b_{ij}}{\det g_{ij}} = \frac{\det(\nabla^2\psi)}{(1 + |\nabla\psi|^2)^{1+(n-1)/2}}. \tag{1.2.6}$$

Example 1. (Model of sphere of radius 1.) A special case of a surface is the sphere

$$S^{n-1}: \; x_n = \pm\sqrt{1-|x'|^2}.$$

Then one can solve the following

**Problem 1.2.2.** *Prove that*

$$K(x) = 1. \tag{1.2.7}$$

The coefficients of the Riemannian metric (1.2.4) are

$$g_{ij} = -\delta_{ij} + (1-|x|^2)^{-1} x_i x_j \tag{1.2.8}$$

according to (1.2.5) and (1.2.7). Using Problem 1.2.1. one can prove

**Problem 1.2.3.** *Prove that*

$$g = \det(g_{ij}) = \frac{1}{x^2+a^2} \tag{1.2.9}$$

*and find the matrix inverse to* $g_{ij}$.

Applications: Set

$$E(u) = \int_{\mathbb{R}^n} \|\nabla u\|^2 dx.$$

This functional is well defined for $u \in C_0^\infty(\mathbb{R}^n)$. The critical points of the functional are such that for any $h \in C_0^\infty(\mathbb{R}^n)$ we have

$$\frac{d}{d\varepsilon}(E(u+\varepsilon h))\,|_{\varepsilon=0} = 0.$$

Since

$$E(u+\varepsilon h) = E(u) + 2\varepsilon \operatorname{Re} \int_{\mathbb{R}^n}\int \langle \nabla u, \nabla h\rangle dx + O(\varepsilon^2),$$

we see that $u$ is a critical point, if and only if

$$\operatorname{Re} \int_{\mathbb{R}^n}\int \langle \nabla u, \nabla h\rangle dx = 0,$$

Applying the Gauss - Green formula we get

$$\operatorname{Re} \int_{\mathbb{R}^n}\int \Delta u h dx = 0, \; \forall h \in C_0^\infty(\mathbb{R}^n).$$

This relation implies

$$\Delta u = 0.$$

**Problem 1.2.4.** *Show that if* $u \in C_0^\infty$ *and* $\Delta u = 0$, *then* $u = 0$.

The functional $E(u)$ defined above satisfies some invariance properties

$$\text{(invariance by translation)}\ \ E(u) = E(\tau_v u),\ \ \tau_v u(x) = u(x+v),\ \ v \in \mathbb{R}^n$$

$$\text{(invariance by rotation)}\ \ E(u) = E(A^* u),\ \ A^* u(x) = u(Ax),\ A \in SO(n).$$

$$\text{(rescaling properties)}\ \ E(S_\lambda u) = \lambda^{(2-n)/2} E(u),\ \ S_\lambda u(x) = u(\lambda x),\ \lambda > 0.$$

The closure of $C_0^\infty$ functions with respect to the norm

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{1/2}$$

is called Dirichlet space $\dot{H}^1(\mathbb{R}^n)$. he closure of $C_0^\infty$ functions with respect to the norm

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} |u|^2\right)^{1/2}$$

is called Sobolev space space $H^1(\mathbb{R}^n)$.

## 1.3 Interpolation

### 1.3.1 Preliminary facts about holomorphic functions

Let **C** be the complex plane and let $U \subseteq \mathbf{C}$ be an open domain in this plane. Any point $z \in U$ can be represented as

$$z = x + iy,$$

where $x, y$ are real numbers. A function

$$f : U \to \mathbf{C}$$

is $C^1(U)$ if the partial derivatives

$$\partial_x f(x+iy), \partial_y f(x+iy)$$

exist and are continuous functions. Of special interest are the vector fields

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

and

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

If $f \in C^1(U)$, then $f$ is called holomorphic in $U$, if satisfies the equation

$$\partial_{\bar{z}} f(z) = 0, \quad z \in U.$$

One can see that a function $f : U \to \mathbf{C}$ is holomorphic in $U$ if and only if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists for any $z \in U$.

The most important formula in the elementary theory of holomorphic functions is the Cauchy theorem and the Cauchy formula.

Let $\Gamma$ be a closed path in $U$ and let $z \in \mathbf{C}$ be a point such that $\Gamma$ does not pass through $z$. Then the index of $z$ with respect to $\Gamma$ is

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}.$$

The Cauchy theorem states that if $\Gamma$ is closed path in $U$ such that $\mathrm{Ind}_\Gamma(w) = 0$ for any $w$ outside $U$, then

$$\int_\Gamma f(\zeta) d\zeta = 0 \tag{1.3.1}$$

for any function holomorphic in $\mathbf{C}$. The corresponding Cauchy formula is

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.3.2}$$

The condition $\mathrm{Ind}_\Gamma(w) = 0$ for $w$ outside $U$ is fulfilled for the case $U$ is simply connected.

Also in the case of a simply connected domain $U$ with smooth boundary $\partial U$ for any function holomorphic in $U$ and continuos in the closure of $U$ we have the corresponding Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.3.3}$$

Applying for example the above formula for $\{z, |z - z_0| < \delta\} \subset U$, we obtain the estimate

$$|\partial_z^k f(z_0)| \leq \frac{Mk!}{\delta^k}, \tag{1.3.4}$$

where

$$M = \sup_{|z-z_0|=\delta} |f(z)|.$$

This estimate guarantees that the formal Taylor series

$$\sum_{k=0}^{\infty} \partial_z^k f(z_0)(z-z_0)^k/k!$$

converges absolutely and uniformly for $|z-z_0|$ sufficiently small and moreover the series coincides with $f(z)$ for $z$ sufficiently close to $z_0$. If $k=0$ the inequality in (1.3.4) gives the maximum principle

$$|f(z_0)| \leq \sup_{|z-z_0|=\delta} |f(z)|,$$

so taking $z_0$ to vary in a fixed disc $|z|<R$ and choosing suitably $\delta>0$ one can solve the following.

**Problem 1.3.1.** *If $f(z)$ is holomorphic in the disc $|z|<R$ and continuous in the closure of the disc, then*

$$\max_{|z|\leq R}|f(z)| = \max_{|z|=R}|f(z)|.$$

### 1.3.2 Il teorema di Riesz-Thorin

**Lemma 1.3.1** (delle tre rette)**.** *Siano date due rette parallele in $\mathbb{C}$, siano esse $l_0$ e $l_1$ e sia $\Omega$ la striscia aperta di piano compresa tra le due rette. Sia $f : \overline{\Omega} \longrightarrow \mathbb{C}$ una funzione continua tale che $f|_\Omega$ sia olomorfa. Supponiamo che*

$$c_0 = \sup_{l_1}|f| < \infty, \qquad c_1 = \sup_{l_1}|f| < \infty$$

*e supponiamo che esistano due costanti reali positive $a$ e $C$ tali che*

$$|f(z)| \leq Ce^{a|z|^2} \qquad \big(\forall z \in \overline{\Omega}\big).$$

*Allora, per ogni retta $l$ parallela a $l_0$ e $l_1$ e tutta contenuta in $\Omega$, esiste un numero $\theta \in (0,1)$ tale che*

$$\sup_l |f| \leq c_0^{1-\theta} c_1^{\theta}.$$

**Dimostrazione.** Osserviamo che, a meno di diffeomorfismo, possiamo supporre

$$l_0 = \{\operatorname{Re} z = 0\} \quad \text{e} \quad l_1 = \{\operatorname{Re} z = 1\}.$$

Iniziamo col provare che, per ogni $\theta \in (0,1)$, posta $l_\theta = \{\operatorname{Re} z = \theta\}$,

$$c_\theta = \sup_{l_\theta}|f| \leq \max\big(c_0, c_1\big).$$

Fissato $a' > a$, si ponga
$$g(z) = f(z)e^{a'z^2}.$$
Se $z = x + iy$ con $x \in [0,1]$ e $|y| \geq 2$, possiamo usare le stime
$$\text{Re}\,(a|z|^2 + a'z^2) = a(x^2 + y^2) + a'(x^2 - y^2) \leq C - (a' - a)y^2.$$
Prendendo $|y| \to \infty$, otteniamo
$$\lim_{|z|\to\infty} |g(z)| \leq \lim_{|z|\to\infty} Ce^{\text{Re}\,(a|z|^2 + a'z^2)} = 0.$$
Si ponga
$$\Omega_R = \left\{z \in \mathbb{C} : 0 \leq \text{ Re } z \leq 1, |\Im z| \leq R\right\}.$$
Denotiamo con $\Gamma_R$ la frontiera di $\Omega_R$. Poiché $\Omega_R$ è compatto, per il teorema di Weierstraß, esiste un punto $z_0$ appartenente a $\Omega_R$ tale che
$$|g(z_0)| = \max_{\Omega_R} |g|.$$
Supponiamo che $z_0$ non appartenga alla frontiera $\Gamma_R$ di $\Omega_R$, ossia $z_0 \in \mathring{\Omega}_R$. Allora esiste una palla $B = \overline{B(z_0, \varrho)}$ massima possibile (ossia tale che $B \cap \Gamma_R \neq \emptyset$) tale che $B \subset \Omega_R$. Per ogni $\bar{r} < \varrho$,
$$g(z) = \sum_{n\geq 0} c_n (z - z_0)^n$$
e, detto $z = z_0 + re^{i\theta}$,
$$g(z) = \sum_{n\geq 0} c_n r^n e^{in\theta}$$
e quindi, per l'identità di Parseval,
$$\sum_{n\geq 0} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z_0 + e^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z_0)|^2 d\theta = |g(z_0)|^2 = |c_0|^2.$$
Pertanto
$$\sum_{n\geq 0} |c_n|^2 r^{2n} \leq |c_0|^2$$
e, facendo il limite per $r \to \varrho$,
$$\sum_{n\geq 0} |c_n|^2 \varrho^{2n} \leq |c_0|^2$$
per cui deve necessariamente essere $c_n = 0$ per ogni $n \geq 1$ e quindi $g$ deve essere costantemente $c_0$ su $B$ cioè, essendo $B \cap \Gamma_R$, deve esistere $z_1 \in \Gamma_R$ tale che
$$g(z_1) = \max_{\Omega_R} |g|.$$

Da ciò segue che

$$|g(z)| \le \max_{w \in \Gamma_R} |g(w)| \qquad \big(\forall z \in \Omega_R\big)$$

e, al limite per $R \to \infty$,

$$|g(z)| \le \max\left(\sup_{l_0} |g|, \sup_{l_1} |g|\right)$$

ed in particolare

$$\sup_{l_\theta} |g| = \sup_{t \in \mathbb{R}} |g(\theta + it)| \le \max\left(\sup_{l_0} |g|, \sup_{l_1} |g|\right).$$

Ricordato, adesso, che $g(z) = f(z)e^{a'z^2}$, si ha

$$|f(\theta)| \le \max\big(c_0, c_1\big)$$

e lo stesso argomento per ogni $y \in \mathbf{R}$ ci da

$$|f(\theta + iy)| \le \max\big(c_0, c_1\big)$$

cosi abbiamo

$$\sup_{l_\theta} |f| = \sup_{t \in \mathbb{R}} |f(\theta + it)| \le \max\big(c_0, c_1\big).$$

Consideriamo, adesso, per ogni $\epsilon > 0$ e ogni $\lambda \in \mathbb{R}$,

$$f_{\epsilon,\lambda}(z) = e^{\epsilon z^2 + \lambda z} f(z).$$

Allora

$$\sup_{l_0} |f_{\epsilon,\lambda}| = \sup_{t \in \mathbb{R}} \left| e^{\epsilon(it)^2 + \lambda(it)} f(it) \right| = \sup_{t \in \mathbb{R}} e^{-\lambda t^2} |f(it)| \le c_0$$

ed inoltre,

$$\sup_{l_1} |f_{\epsilon,\lambda}| = \sup_{t \in \mathbb{R}} \left| e^{\epsilon(1+it)^2 + \lambda(1+it)} f(1+it) \right| = \sup_{t \in \mathbb{R}} e^{\epsilon - \epsilon t^2 + \lambda} |f(+it)| \le e^{\epsilon + \lambda} c_1.$$

Pertanto

$$|f_{\epsilon,\lambda}(\theta + it)| \le \max\left(c_0, c_1 e^{\epsilon + \lambda}\right)$$

ovvero

$$\left| e^{\epsilon(\theta+it)^2 + \lambda(\theta+it)} f(\theta + it) \right| \le \max\left(c_0, c_1 e^{\epsilon + \lambda}\right)$$

e, semplificando l'espressione,

$$e^{\epsilon\theta^2 - \epsilon t^2 + \lambda\theta} |f(\theta + it)| \le \max\left(c_0, c_1 e^{\epsilon + \lambda}\right)$$

che equivale a

$$|f(\theta+it)| \le e^{-\epsilon(\theta^2-t^2)} \max\Big(c_0 e^{-\lambda\theta}, c_1 e^{\epsilon+\lambda(1-\theta)}\Big).$$

Facendo il limite per $\epsilon \to 0$, otteniamo l'espressione,

$$|f(\theta+it)| \le \max\Big(c_0 e^{-\lambda\theta}, c_1 e^{\lambda(1-\theta)}\Big).$$

segliamo $\lambda$ in modo che $c_1 e^{-\lambda\theta} = c_2 e^{\lambda(1-\theta)}$, ovvero, detto $\rho = e^{\lambda}$, vogliamo trovare $\rho$ in modo tale che $c_0\rho^{-\theta} = c_1\rho^{1-\theta}$, ossia $\rho = \frac{c_0}{c_1}$ cioè $\lambda = \log\frac{c_0}{c_1}$. Per questo particolare valore di $\lambda$, il comune valore di $c_0\rho^{-\theta}$ e $c_1\rho^{1-\theta}$ è $c_0^{1-\theta}c_1^{\theta}$ e quindi

$$\sup_{l_\theta} |f| \le c_0^{1-\theta} c_1^{\theta}$$

che prova il lemma. □

**Osservazione 1.3.1.** *L'ipotesi centrale del lemma precedente è il fatto che* $|f(z)| \le Ce^{a|z|^2}$. *Questa ipotesi è essenziale. in effetti, se consideriamo* $f(z) = e^{-\cosh z}$, *sulle rette* $\{\Im z = 0\}$ *e* $\{\Im z = 2\pi\}$ *la funzione è limitata ma sulla retta* $\{\Im z = \pi\}$ *la funzione non è limitata.*

**Problem 1.3.2.** *If*

$$T : L^{p_0} \longrightarrow \mathbf{R}$$

*and* $T : L^{p_1} \to \mathbf{R}$, *then* $T : L^p \to \mathbf{R}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

**Hint** Use Hölder inequality and duality of $L^p$.

**Theorem 1.3.1** (di Riesz-Thorin)**.** *Siano* $(E,\mathscr{E},\mu)$ *e* $(F,\mathscr{F},\nu)$ *due spazi misurati e siano* $p_0, p_1, q_0, q_1 \in [1,\infty]$. *Sia* $T$ *un operatore lineare definito sia su* $L^{p_0}(E)$ *che sue* $L^{p_1}(E)$ *a valori nello spazio delle funzioni misurabili di* $F$ *e tale che*

$$\|Tf\|_{q_0} \le M_0\|f\|_{p_0} \quad e \quad \|Tg\|_{q_1} \le M_1\|g\|_{p_1}$$

*per ogni* $f \in L^{p_0}(E)$ *e ogni* $g \in L^{p_1}(E)$. *Siano* $p$ *e* $q$ *due numeri reali definiti come segue:*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

*dove* $\theta \in (0,1)$. *Allora* $T$ *è definito sullo spazio di Lebesgue* $L^p(E)$ *e*

$$\|Tf\|_q \le M_0^{1-\theta} M_1^{\theta} \|f\|_p.$$

**Dimostrazione.** Supponiamo, inizialmente che $p < \infty$. Lo spazio delle funzioni semplici su $E$ è denso in $L^p(E)$. Pertanto

$$\|T\|_{p,q} = \sup_{f \text{ semplice}} \frac{\|Tf\|_q}{\|f\|_p}, \quad \|Tf\|_q = \sup_{g \text{ semplice}} \frac{\big|\langle Tf, g\rangle\big|}{\|g\|_{q'}}$$

per cui

$$\|T\|_{p,q} = \sup_{f,g \text{ semplici}} \frac{\big|\langle Tf, g\rangle\big|}{\|f\|_p \|g\|_{q'}}.$$

Ne segue che basta dimostrare la disuguaglianza

$$\left|\int_Y Tf(y)g(y)\nu(dy)\right| \le M_0^{1-t} M_1^t \|f\|_p \|g\|_{q'} \tag{1.3.5}$$

quando $f$ e $g$ sono funzioni semplici su $E$ e $F$ rispettivamente, cioè

$$f(x) = \sum_{j\in\mathscr{J}} a_j I_{E_j}(x), \qquad g(y) = \sum_{k\in\mathscr{K}} b_k I_{F_k}(y).$$

Per $0 \le \operatorname{Re} z \le 1$ si definiscano $p_z$ e $q'_z$ come segue:

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1}, \qquad \frac{1}{q'_z} = \frac{1-z}{q'_0} + \frac{z}{q'_1}.$$

Se, per ogni fissato indice $j \in \mathscr{J}$ e $k \in \mathscr{K}$, $a_j = |a_j| e^{i\theta_j}$ e $b_k = |b_k| e^{i\varphi_k}$, si ponga

$$f_z(x) = \sum_{j\in\mathscr{J}} |a_j|^{p/p_z} e^{i\theta_j} I_{E_j}(x), \qquad g_z(y) = \sum_{k\in\mathscr{K}} |b_k|^{q'/q'_z} e^{i\varphi_l} I_{F_k}(y).$$

La funzione $g_z$ è ben definita finché $q' < \infty$. Se $q' = \infty$ allora $q'_0 = q'_1 = \infty$ e quindi $\frac{1}{q'_z} = 0$ per ogni $z$. In tal caso poniamo $\frac{q'}{q'_z} = 1$ per ogni $z$. Si definisca, infine, la funzione

$$F(z) = \int_Y Tf_z(y) g_z(y)\nu(dy) = \sum_{j\in\mathscr{J}, k\in\mathscr{K}} c_{jk} |a_j|^{p/p_z} |b_k|^{q'/q'_z}$$

dove si è posto

$$c_{jk} = e^{i\theta_j} e^{i\varphi_k} \int_Y T\big(I_{E_j}(y)\big) I_{F_k}(y)\nu(dy).$$

Vogliamo provare che la funzione $F$ verifica le ipotesi del *lemma delle tre rette*. Anzitutto $F$ è continua sulla striscia di piano $\overline{\Omega} = \{0 \le \operatorname{Re} z \le 1\}$. Inoltre,

$$|F(z)| \le \sum_{j\in\mathscr{J}, k\in\mathscr{K}} |c_{jk}| \cdot |a_j|^{\operatorname{Re}(p/p_z)} |b_k|^{\operatorname{Re}(q'/q'_z)}$$

per cui $F$ è limitata su $\overline{\Omega}$. Se $t \in \mathbb{R}$, allora

$$|F(it)| \leq M_0 \|f_{it}\|_{p_0} \|g_{it}\|_{q_0'}.$$

Se $p_0 < \infty$, allora

$$\|f_{it}\|_{p_0} = \left( \sum_{j \in \mathscr{J}} |a_j|^p \mu(E_j) \right)^{p_0} = \left(\|f\|_p\right)^{p/p_0}$$

altrimenti, se $p_0 = \infty$, il primo e terzo membro sono eguali ad 1. Analogamente

$$\|g_{it}\|_{q_0'} = \left(\|g\|_{q'}\right)^{q'/q_0'}.$$

Pertanto

$$|F(it)| \leq M_0 \left(\|f\|_p\right)^{p/p_0} \left(\|g\|_{q'}\right)^{q'/q_0'}$$

e, in maniera analoga,

$$|F(1+it)| \leq M_1 \left(\|f\|_p\right)^{p/p_1} \left(\|g\|_{q'}\right)^{q'/q_1}.$$

Per il lemma delle tre rette, dunque,

$$|F(\theta)| \leq M_0^{1-\theta} M_1^{\theta} \left(\|f\|_p\right)^{p\left(\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\right)} \left(\|g\|_{q'}\right)^{q'\left(\frac{1-\theta}{q_0'}+\frac{\theta}{q_1'}\right)} = M_0^{1-\theta} M_1^{\theta} \|f\|_p \|g\|_{q'}.$$

Siccome $f_\theta = f$ e $g_\theta = g$, si ha

$$F(\theta) = \int_Y Tf(y) g(y) \nu(dy)$$

che dimostra la (1.3.5) per $p < \infty$ e $q > 1$. Nel caso in cui $p = \infty$, allora $p_0 = p_1 = \infty$ e, per la disuguaglianza di Hölder,

$$\|Tf\|_q \leq \left(\|Tf\|_{q_0}\right)^{1-\theta} \left(\|Tf\|_{q_1}\right)^{\theta} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_\infty$$

ovvero la tesi. □

Applicazione del teorema di Riesz - Torin implica la disequazione di Young (see [44])

(1.3.6) $$\|f * g\|_{L^q} \leq \|f\|_{L^1} \|g\|_{L^q}$$

for $1 \leq q \leq \infty$. Qui

$$f * g(x) = \int f(x-y) g(y) dy.$$

In fatti, la stima (1.3.12) é ovvia per $q=\infty$ e per $q=1$ e dopo l'interpolazione si ottiene (1.3.12). Si puo generealizzare la seguente versione di (1.3.12)

$$\|f*g\|_{L^s} \le \|f\|_{L^r}\|g\|_{L^p} \tag{1.3.7}$$

per $1/p+1/r=1+1/s$. Per $r=1$ la stima e stata dimostrata in (1.3.12). In particolare, abbiamo

$$\|f*g\|_{L^q} \le \|f\|_{L^1}\|g\|_{L^q} \tag{1.3.8}$$

Per $s=\infty$ la disequazione di Hölder ci da

$$\|f*g\|_{L^\infty} \le \|f\|_{L^{q'}}\|g\|_{L^q}.$$

Interpolazione tra questa stima e (1.3.8) implica (1.3.8).

### 1.3.3 Density of continuous functions in $L^1$

Let $L^q$ denote the Lebesgue space $L^q(\mathbf{R}^n)$.

In this subsection we'll show that continuous functions with compact support are dense in $L^1 = L^1(\mathbb{R}^n, m)$.

The support of a complex valued function $f$ on $\mathbb{R}^n$ is the closure of $\{x \in \mathbb{R}^n : f(x) \neq 0\}$. We'll denote by $C_c(\mathbb{R}^n)$ the set of all complex valued continuous functions on $\mathbb{R}^n$ with compact support.

**Osservazione 1.3.2.** $C_c(\mathbb{R}^n) \subset L^1$

Our goal is to prove

**Theorem 1.3.2.** *For any $f \in L^1$ and any $\varepsilon > 0$ there is a $g$ in $C_c(\mathbb{R}^n)$ such that $\int |f-g|dm < \varepsilon$.*

Thus integrable functions can be approximated by continuous functions. We'll need a special case of Urysohn's Lemma. The proof is deferred to Subsection 1.3.4.

**Lemma 1.3.2.** *Let $X$ be any locally compact metric space, and let Let $K$ and $U$ be subsets of $X$ with $K$ compact, $U$ open, and $K \subset U$. There is a function $\chi \in C_c(X)$ satisfying*

1. *$0 \le \chi(x) \le 1$ for every $x \in X$*
2. *$\chi(x) = 1$ for every $x \in K$*
3. *The support of $\chi$ is a subset of $U$.*

The proof of Theorem 1.3.2 begins with some elementary reductions to simpler special cases. First, since $L^1$ functions can be approximated in $L^1$ by integrable simple functions, it suffices to prove the theorem when $f$ is a simple function. Next, since an integrable simple function is a linear combination of functions of the form $\chi_E$, where $E$ is a measurable set of finite measure, it suffices to prove the theorem when $f = \chi_E$. Therefore, Theorem 1.3.2 will be proved by establishing

**Proposition 1.3.1.** *Let $E$ be a measurable subset of* $\mathrm{R}^n$ *with finite measure. Then for any $\varepsilon > 0$ there is a $g \in C_c(\mathbb{R}^n)$ with $\int \left|\chi_E - g\right| dm < \varepsilon$*

*Proof.* By regularity properties of Lebesgue measure, there are a compact set $K$ and an open set $U$ such that $K \subset E \subset U$ and $m(U \backslash K) < \varepsilon$. By Lemma 1.3.2, there is a $g \in C_c(\mathbb{R}^n)$ such that $0 \le g \le 1$ everywhere, $g = 1$ on $K$, and $g$ vanishes outside of $U$. It follows that $\left|g - \chi_E\right|$ vanishes outside of $U \backslash K$, and that $\left|g - \chi_E\right| \le 1$ on $U \backslash K$. Therefore

$$\int \left|g - \chi_E\right| dm = \int_{U \backslash K} \left|g - \chi_E\right| dm \le m(U \backslash K) < \varepsilon$$

□

Let us recall Lusin's Theorem: Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable, , and suppose that $f$ vanishes outside a set of finite measure. Then for any $\varepsilon > 0$ there is a measurable set $E$ with $m(E) < \varepsilon$ such that $f\big|_{\mathrm{R}^n \backslash E}$ is continuous.

### 1.3.4 Proof of Urisohn lemma

In this subsection, we'll prove the version of Urysohn's Lemma that we used in the proof of Theorem 1.3.2. We'll work in a locally compact metric space $(X, d)$ We'll use the notation $B_r(x)$ for the open ball in $X$ with center $x$ and radius $r$

**Lemma 1.3.3.** *Let $x_0 \in X$ and $r > 0$. There is a $\chi \in C_c(X)$ with*

1. $0 \le \chi(x) \le 1$ *for every* $x \in X$
2. $\chi(x) = 1$ *for every* $x \in B_r(x_0)$
3. *The support of* $\chi$ *is a subset of* $B_{2r}(x_0)$.

*Proof.* Let $\varphi$ be the continuous function on $\mathbb{R}$ defined by $\varphi(t) = 1$ for $0 \le t \le r$ $\varphi(t) = 1 - \frac{2}{r}(t - r)$ for $r < t \le \frac{3r}{2}$, and $\varphi(t) = 0$ for $t > \frac{3r}{2}$. Let $\chi(x) = \varphi(d(x, x_0))$
□ □

*Proof of Lemma 1.3.2.* Cover $K$ by finitely many balls $B_j = B_{r_f}(x_j)$ such that $\overline{B_{2r_j}(x_j)}$ is a compact subset of $U$. Let $\chi_j$ be the function obtained from Lemma 1.3.3 with $x_0 = x_j$ and $r = r_j$. Let $\varphi = \sum \chi_j$. Then

1. $\varphi$ is continuous on $X$;
2. The support of $\varphi$ is a compact subset of $U$;
3. $\varphi \geq 0$ everywhere, and $\varphi \geq 1$ on $K$

Let $\chi = \min\{\varphi, 1\}$. Then $\chi$ has all the required properties. □

### 1.3.5 Basic interpolation theorems

The first important interpolation theorem is the Riesz-Thorin interpolation theorem. To state this theorem we start with some notations.

Given any positive real numbers $p_0, p_1$ with $1 \leq p_0 < p_1 \leq \infty$, we denote by $L^{p_0}(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$ the linear space

$$\{f : f = f_0 + f_1, f_0 \in L^{p_0}(\mathbf{R}^n), f_1 \in L^{p_1}(\mathbf{R}^n)\}.$$

The norm in this space we define as follows

$$\|f\|_{L^{p_0}+L^{p_1}} = \inf_{f=f_0+f_1} \|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}.$$

Here the infimum is taken over all representations $f = f_0 + f_1$, where $f_0 \in L^{p_0}(\mathbf{R}^n)$ and $f_1 \in L^{p_1}(\mathbf{R}^n)$.

It is easy to see that $L^{p_0} + L^{p_1}$ is a Banach space.

**Theorem 1.3.3.** *(Riesz - Torin) Suppose $T$ is a linear bounded operator from $L^{p_0} + L^{p_1}$ into $L^{q_0} + L^{q_1}$ satisfying the estimates*

$$\|Tf\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}, \; f \in L^{p_0},$$
$$\|Tf\|_{L^{q_1}} \leq M_0 \|f\|_{L^{p_1}}, \; f \in L^{p_1}. \tag{1.3.9}$$

*Then for any $t \in (0,1)$ we have*

$$\|Tf\|_{L^{q_t}} \leq M_0 \|f\|_{L^{p_t}}, \tag{1.3.10}$$

*where*

$$1/p_t = t/p_1 + (1-t)/p_0 \;\; , \;\; 1/q_t = t/q_1 + (1-t)/q_0. \tag{1.3.11}$$

Applying this interpolation theorem, one can derive (see [44]) the Young inequality

$$\|f * g\|_{L^q} \leq \|f\|_{L^1} \|g\|_{L^q} \tag{1.3.12}$$

for $1 \le q \le \infty$. Here

$$f * g(x) = \int f(x-y)g(y)dy.$$

It is not difficult to derive the following more general variant of (1.3.12)

$$\|f * g\|_{L^s} \le \|f\|_{L^r}\|g\|_{L^p} \tag{1.3.13}$$

for $1/p + 1/r = 1 + 1/s$.

Further, we turn to a weighted variant of Young inequality. For simplicity, we consider only the continuous case. Let $w(x), w_1(x)$ and $w_2(x)$ be smooth positive functions satisfying the assumption

$$w(x+y) \le C w_1(x) w_2(y). \tag{1.3.14}$$

Then the argument of the proof of Young inequality leads to

$$\|w(f * g)\|_{L^q} \le C\|w_1 f\|_{L^1}\|w_2 g\|_{L^q} \tag{1.3.15}$$

Indeed, we have the inequality

$$|w(x)(f * g)(x)| \le C(|w_1 f| * |w_2 g|)(x)$$

and (1.3.15) follows from the classical Young inequality.

Two typical examples of weights satisfying the assumption (1.3.14) are considered below.

**Example 1.** let $w(x) =< x >^s$ with $s > 0$. Then we can choose $w_1 = w_2 = w$ and the assumption (1.3.14) is fulfilled.

**Example 2.** Let $w(x) =< x >^s$ with $s < 0$. Then we take $w_1(x) =< x >^{-s}$ and $w_2(x) =< x >^s$ . Again (1.3.14) is fulfilled.

To prove the Sobolev inequality we need more fine interpolation theorems concerning the weak $L^p$ spaces. To define these weak spaces we shall denote by $\mu$ the Lebesgue measure. Given any measurable function $f$ we shall say that $f \in L^p_w$ if the quantity

$$\sup_t \left(t^p \mu\{x : |f(x)| > t\}\right)^{1/p} \tag{1.3.16}$$

is finite. Note that the quantity in (1.3.16) is not a norm. This quantity is equivalent to

$$\|f\|_{L^p_w} = \sup_{A,\ \mu(A)<\infty} \mu(A)^{-1/p'} \int_A |f(x)|dx,\ \frac{1}{p} + \frac{1}{p\prime} = 1. \tag{1.3.17}$$

**Problem.** Show that the quantities in (1.3.16) and (1.3.17) are equivalent.

We have the inclusion $L^p \subset L^p_w$ in view of the inequality

$$\|f\|^p_{L^p} \geq \int_{|x|\geq t} |f(x)|^p dx \geq t^p \mu\{x : |f(x)| > t\} \sim \|f\|^p_{L^p_w}.$$

**Example.** The function $|x|^{-n/p}$ is in $L^p_w$, but not in $L^p$.

The following two theorems play crucial role in the interpolation theory.

**Theorem 1.3.4.** (**Marcinkiewicz interpolation theorem**) *Suppose $T$ is a linear operator satisfying the estimates*

$$\begin{aligned} \|Tf\|_{L^{q_0}_w} &\leq M_0 \|f\|_{L^{p_0}} \\ \|Tf\|_{L^{q_1}_w} &\leq M_0 \|f\|_{L^{p_1}} \end{aligned} \tag{1.3.18}$$

*with $p_0 \neq p_1$, $1 \leq p_0 \neq p_1 \leq \infty$ and $1 \leq q_0 \neq q_1 \leq \infty$.*

*Then we have*

$$\|Tf\|_{L^q} \leq M_0 \|f\|_{L^p}, \tag{1.3.19}$$

*provided*

$$1/p = t/p_1 + (1-t)/p_0 \ , \ 1/q = t/q_1 + (1-t)/q_0 \tag{1.3.20}$$

*for some $t \in (0,1)$ and $p \leq q$.*

**Theorem 1.3.5.** (**Hunt interpolation theorem**) *Suppose $T$ is a linear operator satisfying the inequalities*

$$\begin{aligned} \|Tf\|_{L^{q_0}} &\leq M_0 \|f\|_{L^{p_0}} \\ \|Tf\|_{L^{q_1}} &\leq M_0 \|f\|_{L^{p_1}} \end{aligned} \tag{1.3.21}$$

*with $1 \leq p_1 < p_0 \leq \infty$ and $1 \leq q_1 < q_0 \leq \infty$. Then for any $t \in (0,1)$ we have*

$$\|Tf\|_{L^{q_t}_w} \leq M_0 \|f\|_{L^{p_t}_w}, \tag{1.3.22}$$

*where*

$$1/p_t = t/p_1 + (1-t)/p_0 \ , \ 1/q_t = t/q_1 + (1-t)/q_0. \tag{1.3.23}$$

As an application of the above interpolation theorems one can prove (see [44]) the following generalization of the Young inequality

$$\|f * g\|_{L^s} \leq \|f\|_{L^p} \|g\|_{L^r_w} \tag{1.3.24}$$

for $1/p + 1/r = 1 + 1/s, 1 < p, r, s < \infty$.

After this preparation we can turn to the proof of the following Sobolev estimate.

**Lemma 1.3.4.** *Suppose* $0<\lambda<n, f\in L^p(\mathbf{R}^n), g\in L^r(\mathbf{R}^n)$, *where* $1/p+1/r+\lambda/n=2$ *and* $1<p,r<\infty$. *Then we have*

$$\int\int \frac{|f(x)||g(y)|}{|x-y|^{\lambda}}dxdy \le C\|f\|_{L^p}\|g\|_{L^r} \tag{1.3.25}$$

**Proof of Lemma 1.3.4** We know that (1.3.24) is fulfilled. Then for the left hand side of the Sobolev inequality (1.3.25) we can apply the Hölder inequality so we get

$$\int\int \frac{|f(x)||g(y)|}{|x-y|^{\lambda}}dxdy \le C\|f\|_{L^p}\|g*h\|_{L^{p'}} \tag{1.3.26}$$

with $h(x)=|x|^{-|\lambda|}$. Now the application of (1.3.24) yields

$$\|g*h\|_{L^{p'}} \le \|g\|_{L^r}\|h\|_{L^l_w} \tag{1.3.27}$$

provided

$$\frac{1}{p'}+1=\frac{1}{r}+\frac{1}{l} \tag{1.3.28}$$

The example considered after the definition of the weak $L^p$ spaces shows that the quantity $\|h\|_{L^l_w}$ is bounded when $\lambda l=n$. From this relation and (1.3.28) we see that for $2=1/p+1/r+\lambda/n$ we have the Sobolev inequality.

### 1.3.6 Idea of abstract interpolation: Interpolation couples

Let $A_0$ and $A_1$ are Banach spaces. Set $\overline{A}=(A_0,A_1)$. We shall call $A_0$ and $A_1$ compatible if there exists separable topological vector space $\mathfrak{U}$, such that $A_0\cup A_1\subset\mathfrak{U}$. We can define

$$\Delta(\overline{A})=A_0\cap A_1, \tag{1.3.29}$$

while for any compatible couple $\overline{A}=(A_0,A_1)$ we can define

$$\Sigma(\overline{A})=A_0+A_1=\{a=a_0+a_1, a_0\in A_0, a_1\in A_1\}. \tag{1.3.30}$$

**Lemma 1.3.5.** *We have the properties:*

- $\Delta(\overline{A})$ *is a Banach space with norm* $\|a\|_{A_0}+\|a\|_{A_1}$;
- $\Sigma(\overline{A})$ *is a Banach space with norm*

$$\|a\|_{\Sigma(\overline{A})}=\inf\{\|a_0\|_{A_0}+\|a_1\|_{A_1}; a=a_0+a_1, a_0\in A_0, a_1\in A_1\}.$$

**Definizione.** *Let* $A_0$ *and* $A_1$ *are Banach spaces and* $\overline{A} = (A_0, A_1)$. *The space* $F(\bar{A})$ *consists of all functions*

$$f(z) : S = \{0 \le \mathrm{Re} z \le 1\} \to \Sigma(\bar{A})$$

*defined, continuous and bounded in the closed strip*

$$S = \{0 \le \mathrm{Re} z \le 1\},$$

*analytic in the open strip*

$$S_0 = \{0 < \mathrm{Re} z < 1\},$$

*and satisfying*

$$\lim_{y \to \infty} f(j + iy) = 0, \quad j = 0, 1.$$

The interpolation space $(A_0, A_1)_\theta$ for $\theta \in [0,1]$ consists of all $a \in \Sigma(\bar{A})$ such that $a = f(\theta)$ for some $f \in F(\bar{A})$. The corresponding norm is defined as follows

$$\|a\|_\theta = \inf\{\|f\|_F ; a = f(\theta), f \in F(\bar{A})\}.$$

It is clear that $(A_0, A_1)_\theta$ is a Banach space.

One can show that

(1.3.31) $$A_0 \cap A_1 \text{ is dense in } (A_0, A_1)_\theta.$$

Moreover, for $f \in A_0 \cap A_1$ we have $f \in (A_0, A_1)_\theta$ and the following estimate

(1.3.32) $$\|f\|_{(A_0,A_1)_\theta} \le C \|f\|_{A_0}^{1-\theta} \|f\|_{A_1}^{\theta}$$

is fulfilled.

The next Theorem gives an estimate of the norm of a bounded operator with respect to interpolation space.

**Theorem 1.3.6.** *Let* $(A_0, A_1)$ *and* $(B_0, B_1)$ *be interpolation couples and let* $T$ *be a bounded operator from* $A_0 + A_1$ *into* $B_0 + B_1$, *such that* $T \in L(A_j, B_j)$ *with norm* $\|T\|_{L(A_j,B_j)}$ *for* $j = 0, 1$. *Then for any* $\theta, 0 < \theta < 1$ *we have*

$$T \in L((A_0, A_1)_\theta, (B_0, B_1)_\theta))$$

*with*

$$\|Tf\|_{(B_0,B_1)_\theta} \le \|T\|_{L(A_0,B_0)}^{1-\theta} \|T\|_{L(A_1,B_1)}^{\theta} \|f\|_{(A_0,A_1)_\theta}.$$

**Proof.** Let $f \in (A_0, A_1)_\theta$. Then there exists a function $f(z) \in F((A_0, A_1))$ so that $f = f(\theta)$. Consider the function

$$g(z) = \|T\|^{z-\theta}_{L(A_0,B_0)} \|T\|^{-z+\theta}_{L(A_1,B_1)} Tf(z).$$

Then $g(z) \in F(B_0, B_1)$. Since

$$\|g(it)\|_{B_0} \le \|T\|^{-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|T\|_{L(A_0,B_0)} \|f(it)\|_{A_0}$$

and

$$\|g(1+it)\|_{B_0} \le \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{-1+\theta}_{L(A_1,B_1)} \|T\|_{L(A_1,B_1)} \|f(it)\|_{A_1},$$

we see that

$$\|Tf\|_{(B_0,B_1)_\theta} \le \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|f\|_{(A_0,A_1)_\theta}.$$

This completes the proof.

A trivial modification in the above proof shows that we have the following.

**Theorem 1.3.7.** *Let $(A_0, A_1)$ and $(B_0, B_1)$ be interpolation couples and let $T(z)$ be a holomorphic in $S_0$ operator-valued function defined in the strip $S$ and continuous there. Suppose that for $z \in S$ we have that $T(z)$ is a linear bounded operator from $A_0 + A_1$ into $B_0 + B_1$, such that $T(j+it) \in L(A_j, B_j)$ with norm*

$$\sup_{t\in\mathbf{R}} \|T(j+it)\|_{L(A_j,B_j)} < \infty$$

*for $j = 0, 1$. Then for any $\theta, 0 < \theta < 1$ we have*

$$T(\theta) \in L((A_0, A_1)_\theta, (B_0, B_1)_\theta)).$$

## 1.4 Idea to define Distributions, why they are needed?

The purpose of this section is to recall some basic notions and properties of the spaces , where the solutions of the PDE are defined.

The first important space is $C_0^\infty(\mathbf{R}^n)$. This space consists of all smooth functions with compact support.

The space $C_0^\infty(\mathbf{R}^n)$ is nonempty linear vector space. Indeed, we can first construct a smooth function $f(x)$, such that $f(x) = 0$ for $x \le 0$ and $f(x) > 0$ for $x > 0$. For the purpose take

$$f(x) = e^{-1/x}$$

for $x > 0$. Then $f(x)f(1-x)$ is a smooth function with support in the interval $[0,1]$.

Recall that the support of function $f(x)$ defined for $x \in \mathbf{R}^n$ is the closure of the set

$$\{x : f(x) \neq 0\}.$$

Sometimes this space is called space of test functions and is denoted by $D(\mathbf{R}^n)$. This space can be equipped with infinite number of semi norms. In fact, given any integer $N \ge 1$ we define

$$\|f\|_N = \max\{|\partial^\alpha f(x)|; |\alpha| \le N\}, \tag{1.4.33}$$

where here and below $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

In order to work with a complete space, i.e. space where any Cauchy sequence converges to an element of the space, we have to define the topology on $C_0^\infty(\mathbf{R}^n)$ using all the collection of semi norms $\|.\|_N$. To define a complete topology the simplest way is to define the convergence of a sequence of functions $\{f_k\}_{k=1}^\infty$ to zero. Recall that this sequence converges to zero if there exists a compact set $K$ such that $\mathrm{supp} f_k \subseteq K$ for any integer $k \ge 1$ and $\|f_k\|_N$ tends to 0 as $k \to \infty$ for any integer $N \ge 0$. Applying the Arzela-Ascoli compactness theorem one can check that the topology determined by this convergence is complete.

We refer to [45] for a complete discussion of the topology on this space.

In a similar way one can consider the space $C^\infty(\mathbf{R}^n)$ consisting of all smooth functions. Now we have the following family of **semi norms**.

$$\|f\|_N = \max\{|\partial^\alpha f(x)|; |x| \le N, |\alpha| \le N\},$$

The above family of semi norms enables one directly to introduce a complete topology ( even one can introduce a complete metric).

The weakest space, where we shall look for solutions of the nonlinear partial differential equations, is the space of distributions $D'(\mathbf{R}^n)$ consisting of all linear continuous functionals on $C_0^\infty(\mathbf{R}^n)$. Given any distribution $\Lambda$ we shall denote by

$$< \Lambda, f >$$

the action of the distribution (the linear functional) $\Lambda$ on the test function $f \in C_0^\infty(\mathbf{R}^n)$. It is clear that

$$C_0^\infty(\mathbf{R}^n) \subset D'(\mathbf{R}^n)$$

and

$$< \Lambda, f >= \int \Lambda(x) f(x) dx$$

for $\Lambda \in C_0^\infty(\mathbf{R}^n)$.

A typical example of a distribution, which is not a test function, is the Dirac delta function $\delta$ defined by

$$< \delta, f >= f(0).$$

Since the space of distributions is the dual space to the space of test functions, we choose the topology on the space of distributions to be the weak topology on this dual space. This means that a sequence of distributions $\{\Lambda_k\}_{k=1}^\infty$ tends to zero if for any test function $f$ we have $< \Lambda_k, f >$ tends to zero.

The space $D'(\mathbf{R}^n)$ equipped with this weak topology is a complete space.

Another useful characterization of the distributions is the following one. A linear functional $\Lambda$ on $C_0^\infty(\mathbf{R}^n)$ is bounded if for any compact $K \subseteq \mathbf{R}^n$ there exist integer $k \geq 0$ and a positive real number $C$ so that for any smooth function $\varphi(x)$ with compact support in $K$ we have

$$| < \Lambda, \varphi > | \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi(x)|.$$

**Example.** Let $\varphi(x)$ be smooth non-negative function such that $\varphi(0) > 0$. Given any $\varepsilon > 0$, we can define the function

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon). \tag{1.4.34}$$

Then it is easy to see that

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon = c\delta,$$

where

$$c = \int \varphi(x) dx > 0.$$

Another fact for distributions is the meaning of the identity

$$\Lambda = 0$$

in the sense of distributions. This means

$$< \Lambda, \varphi >= 0$$

for any test function $\varphi$.

A natural operation in the space of distribution is the differentiation defined by

$$< \partial^\alpha \Lambda, f >= (-1)^{|\alpha|} < \Lambda, \partial^\alpha f > .$$

Then $\partial^\alpha \Lambda$ is a bounded linear functional on $C_0^\infty(\mathbf{R}^n)$ provided $\Lambda \in D'(\mathbf{R}^n)$.

**Problem 1.4.1.** *Given a distribution* $\Lambda$ *on* $\mathbf{R}$ *with*

$$\Lambda' = 0$$

*in the sense of distributions, find a constant* $c$ *such that*

$$\Lambda = c$$

*also in the sense of distributions.*

A distribution $u \in S'(\mathbb{R}^n)$ is called non negative ( $u \geq 0$) if $< u, \varphi >\geq 0$ for any $\varphi \in S(\mathbb{R}^n)$.

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